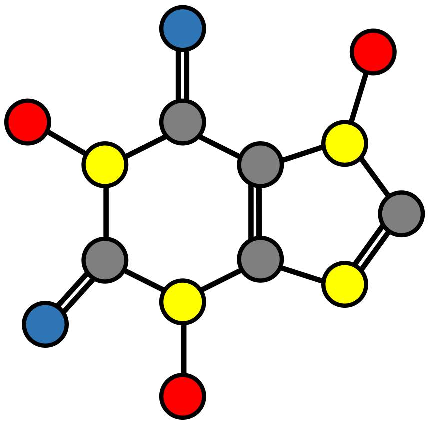

Groups & Homogeneous Spaces
Michael Bronstein – Geometric Deep Learning – Oxford 2024

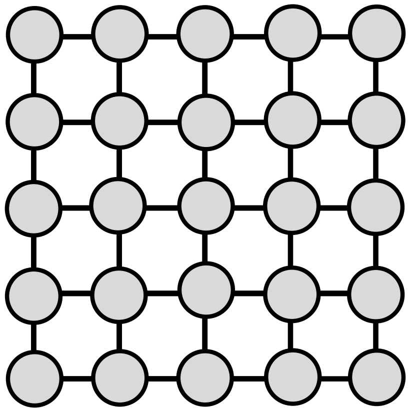
Outline

- Spatial generalisation of convolution: *Group convolution* = transform + match
- Homogeneous spaces have “global” symmetry structure: the group acts transitively
- Homogeneous spaces are equivalent to quotient spaces, where we ”factor out” the stabiliser group
- Regular vs Induces representations and steerability
- Group convolutions are universal linear equivariants
- Next lectures: applications of these ideas to Geometric Graphs *&* extensions to Manifolds, where we need to act locally

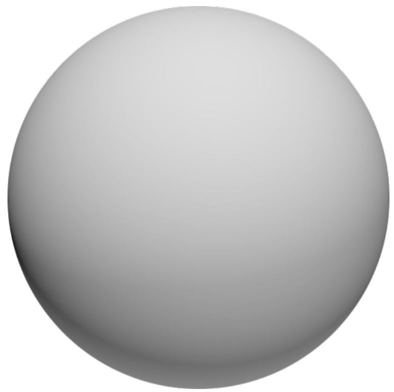
Different Domains



Sets & Graph



Grids



Homogeneous spaces

Manifolds, Meshes & Geometric graphs

Popular architectures as instances of the Blueprint

Architecture	**Domain Ω**	**Symmetry Group G**
CNN	Grid	Translation
Spherical CNN	Sphere / $\mathrm{SO}(3)$	Rotation $\mathrm{SO}(3)$
Intrinsic / Mesh CNN	Manifold / Mesh	Isometry $\mathrm{Iso}(\Omega)$ / Gauge Symmetry $\mathrm{SO}(2)$
GNN	Graph	Permutation Σ_n
Deep Sets	Set	Permutation Σ_n
Transformer	Complete Graph	Permutation Σ_n
LSTM	1D Grid	Time warping

GROUP CONVOLUTION

Convolution, revisited

Convolution, revisited

convolution = matching shifted filter

$$(x \star \psi)(u) = \langle x, S_u \psi \rangle = \int_{-\infty}^{+\infty} x(v)\psi(u - v)\mathrm{d}v$$

shift vector

shift operator

domain $\Omega \cong$ symmetry group G

i.e., $\star\,\psi \colon \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$

Group Convolution

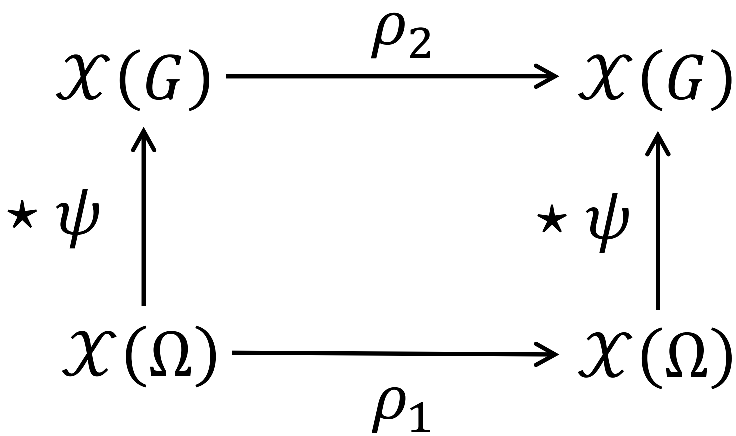
convolution = matching transformed filter

$$(x \star \psi)(g) = \langle x, \rho(g)\psi \rangle = \int_{\Omega} x(v)\psi(g^{-1}v)\mathrm{d}v$$

The convolution outputs a signal on the group G

i.e., $\star\,\psi: \mathcal{X}(\Omega) \to \mathcal{X}(G)$

Group Convolution Equivariance

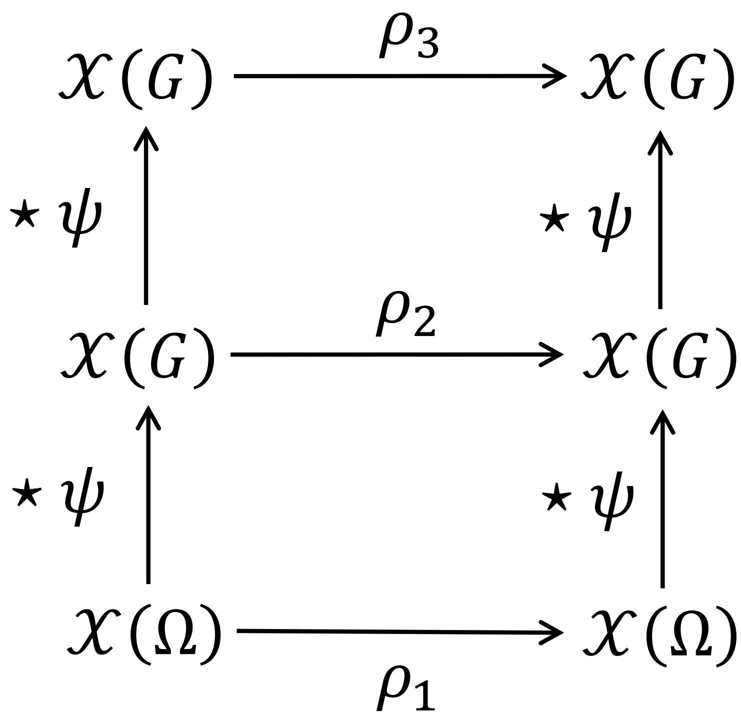


Regular representation

$$\rho_2(g)x(h) = x(g^{-1}h)$$

$$\rho_1(g)x(u) = x(g^{-1}u)$$

Group Convolution Equivariance



Regular representation

$$\rho_2(g)x(h) = x(g^{-1}h)$$

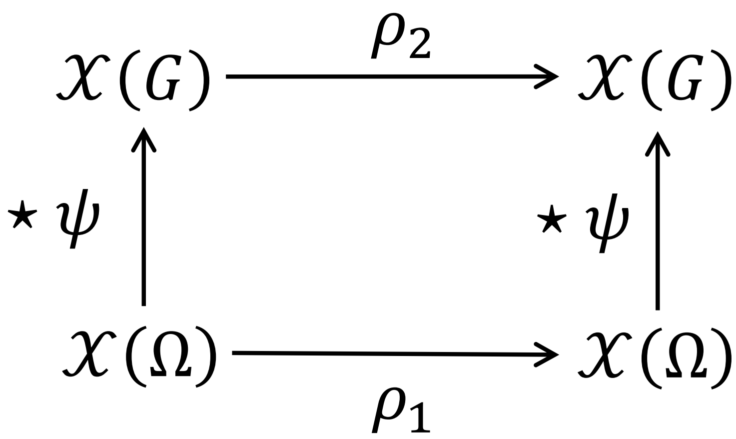
$$\rho_1(g)x(u) = x(g^{-1}u)$$

Note: integration on the group is done w.r.t. the **Haar measure**.

Group Convolution Equivariance

Group convolution is G-equivariant: $\big((\rho_1(h)\,x) \star \psi\big)(g) = \rho_2(h)(\psi \star x)(g)$

Exercise: prove.

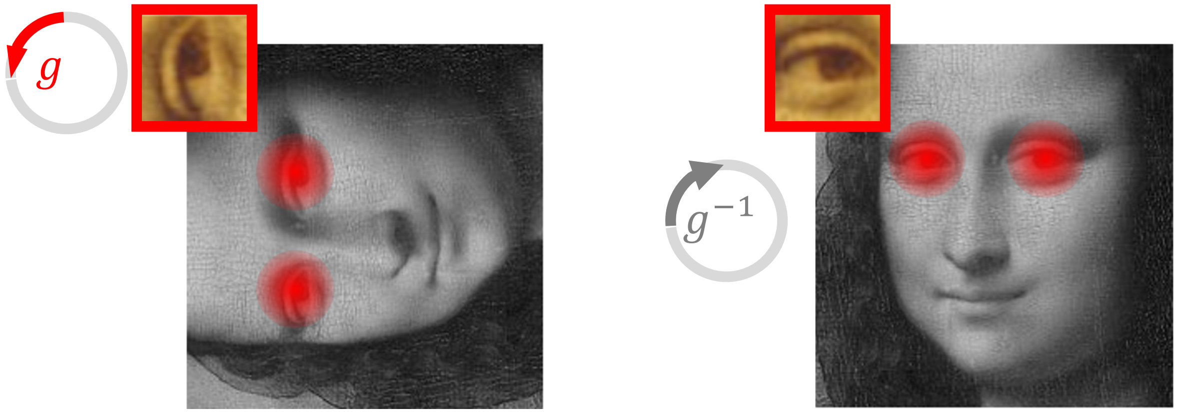


Regular representation

$$\rho_2(g)x(h) = x(g^{-1}h)$$

$$\rho_1(g)x(u) = x(g^{-1}u)$$

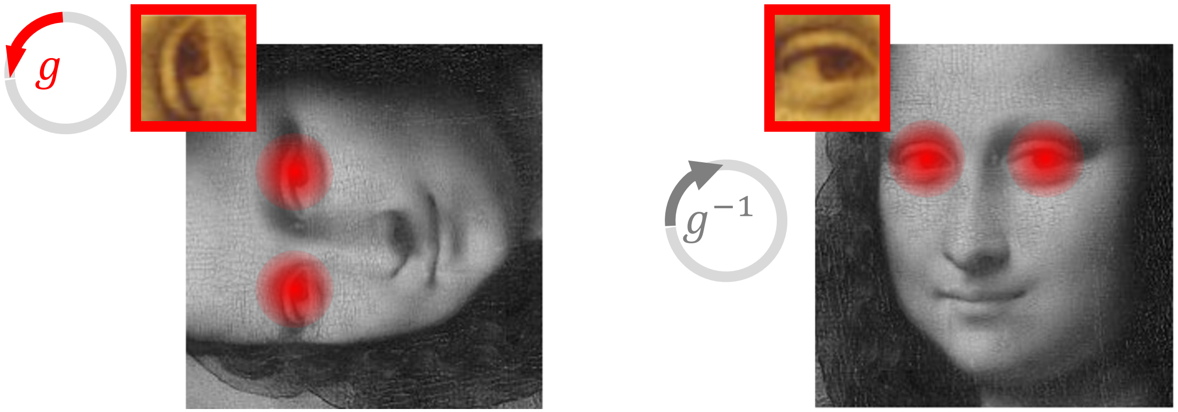
Group Convolution Equivariance



$$\langle x, \rho_1(g)\psi \rangle \quad = \quad \langle \rho_1(g^{-1})x, \psi \rangle$$

Matching a signal with transformed filter = matching inverse-transformed signal with untransformed filter

Group Convolution Equivariance

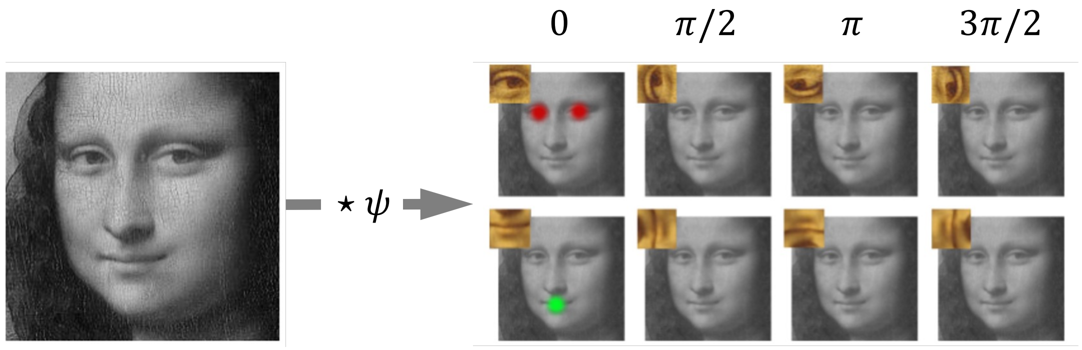


$$\langle x, \rho_1(g)\psi \rangle \quad = \quad \langle \rho_1(g^{-1})x, \psi \rangle$$

Proof of equivariance:

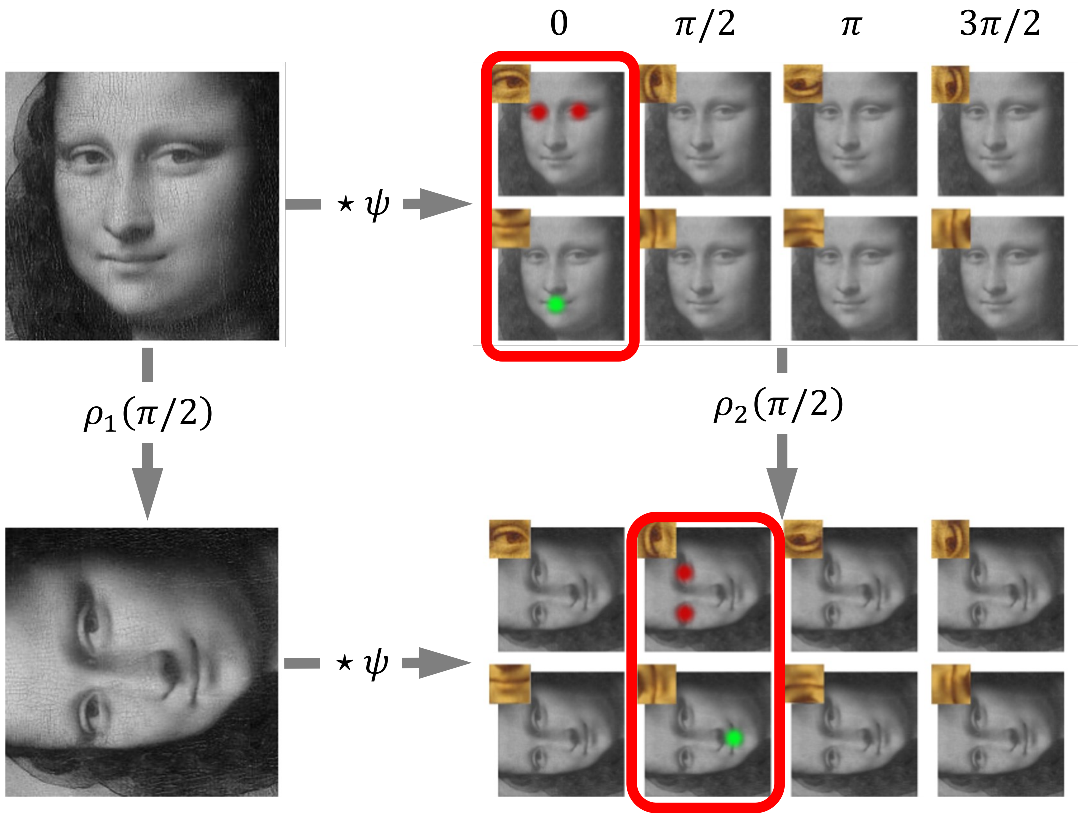
$$
\begin{aligned}
((\rho_1(h)\, x) \star \psi)(g) &= \langle \rho_1(h)x, \rho_1(g)\psi \rangle \\
&= \langle x, \rho_1(h^{-1})\rho_1(g)\psi \rangle \\
&= \langle x, \rho_1(h^{-1}g)\psi \rangle \\
&= \psi \star x(h^{-1}g) \\
&= \rho_2(h)(\psi \star x)(g)
\end{aligned}
$$

Convolution with discrete roto-translations



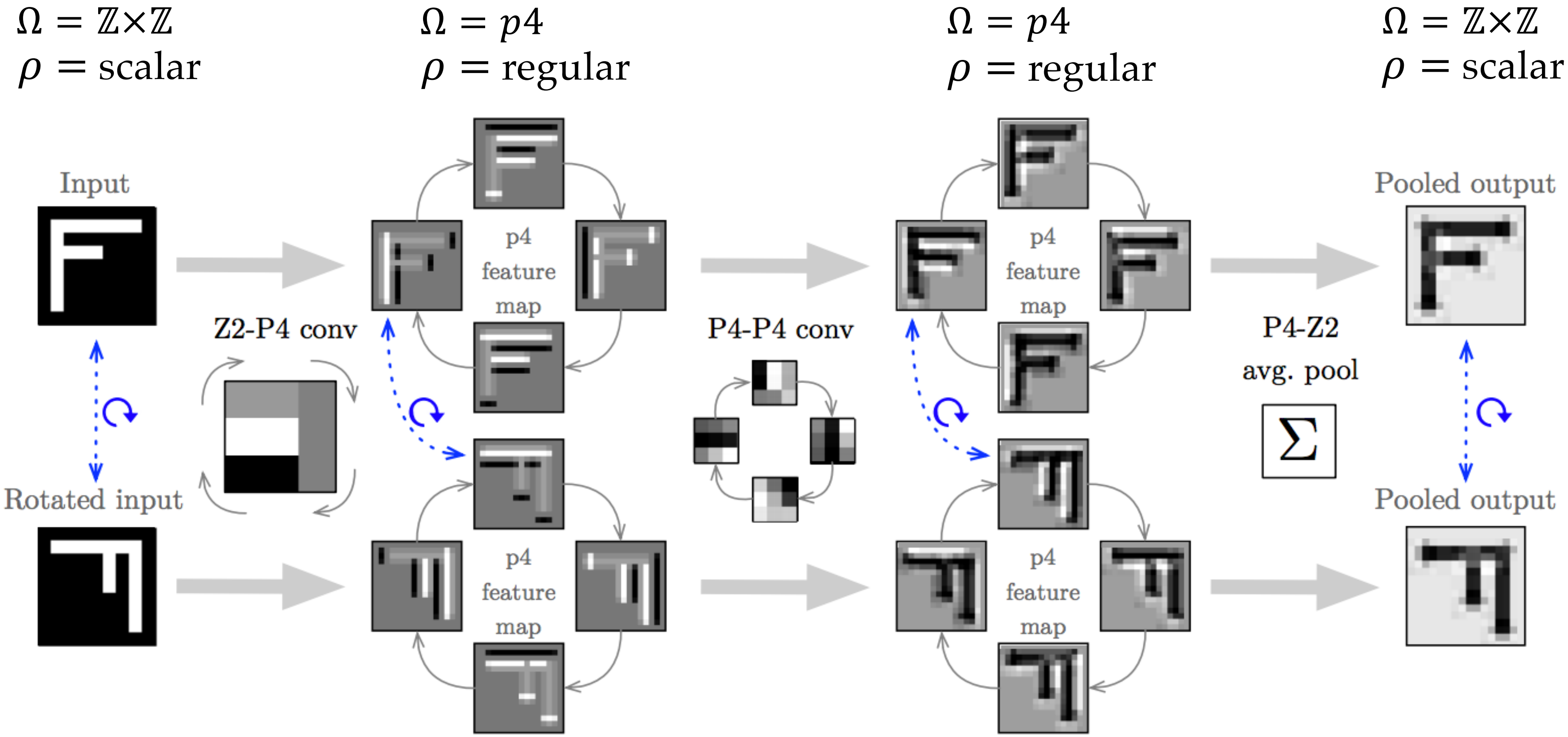
Cohen, Welling 2016

Convolution with discrete roto-translations



Cohen, Welling 2016

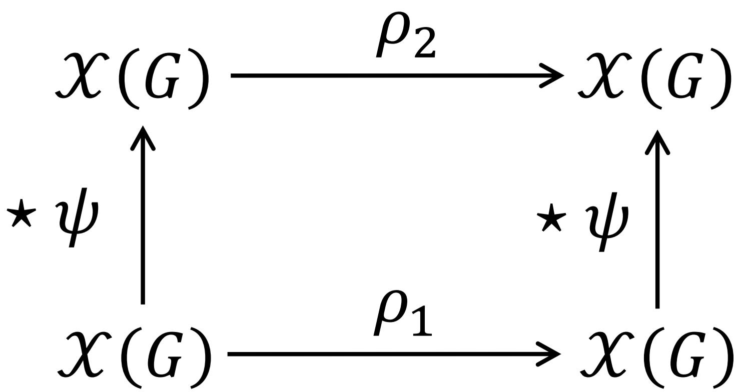
Group Convolution Equivariance



Veeling et al. 2018

"Convolution is all you need"

Theorem: any linear equivariant map between regular representations is a group convolution.



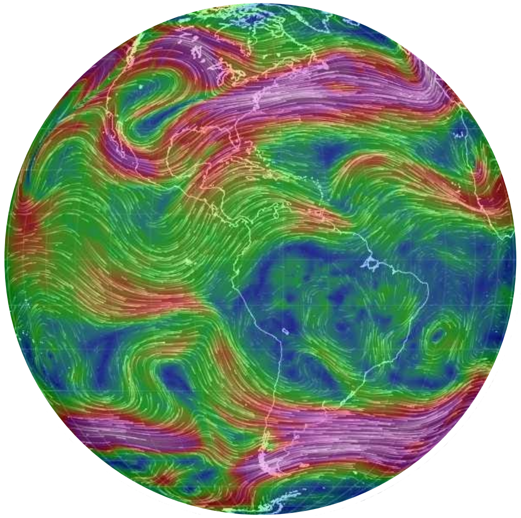
Note: This can be considered an extension of the result we have seen before, that any linear translation equivariant is a (standard) convolution.

Kondor, Trivedi 2018; Cohen et al. 2019, 2021; Bekkers 2020; Aronsson 2021

SPHERICAL CNN

Why sphere?

Astronomy



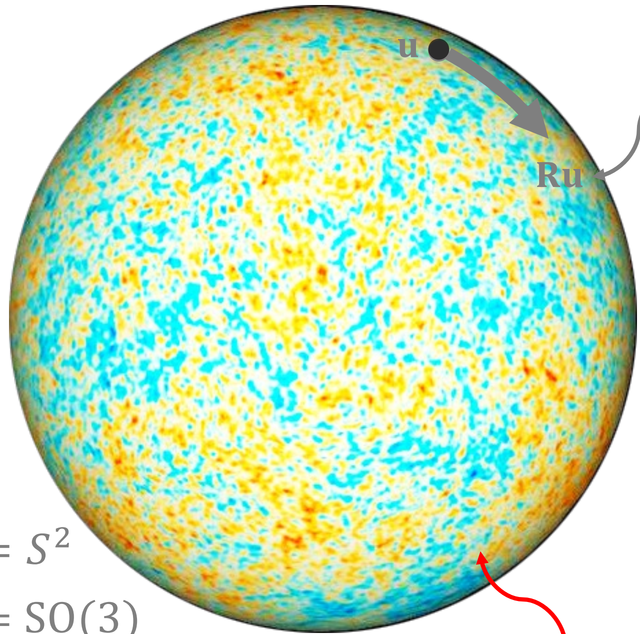
Earth sciences

Omnidirectional vision

Convolution on the Sphere

$$(x \star \psi)(\mathbf{R}) = \int_{S^2} x(\mathbf{u})\psi(\mathbf{R}^{-1}\mathbf{u})d\mathbf{u}$$

signal on SO(3)

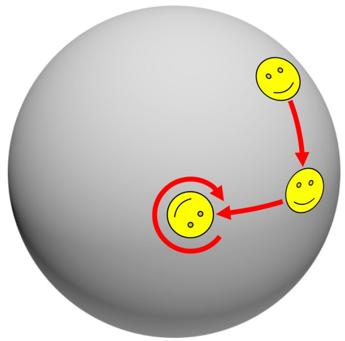


rotation
transformation

sphere $\Omega = S^2$

rotation group $G = \mathrm{SO}(3)$

spherical signal x

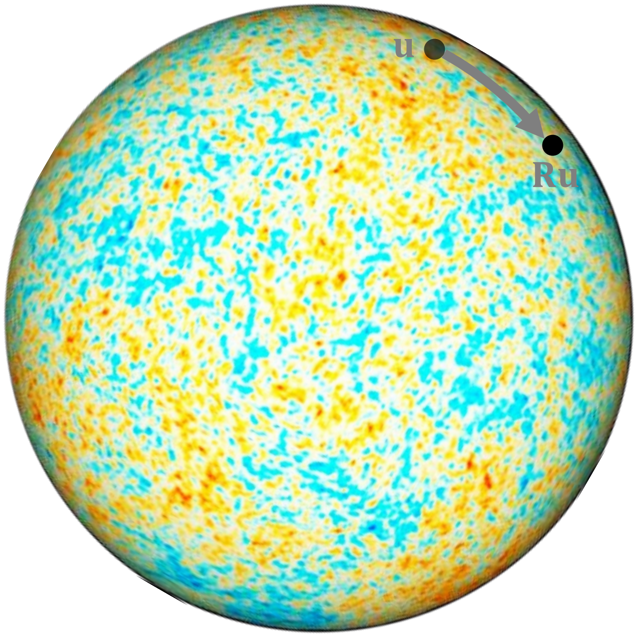


Cohen, Welling 2018

Convolution on the Sphere

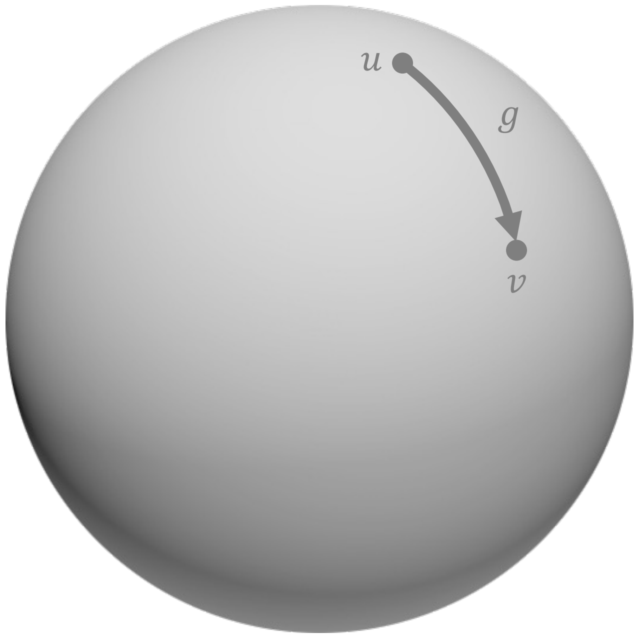
$$((x \star \psi) \star \phi)(\mathbf{R}) = \int_{\mathrm{SO}(3)} (x \star \psi)(\mathbf{Q})\phi(\mathbf{R}^{-1}\mathbf{Q})\mathrm{d}\mathbf{Q}$$

Convolution on SO(3)



Cohen, Welling 2018

Homogeneous Spaces



transitive group action

$$\exists g \in G \text{ s.t. } gu = v$$

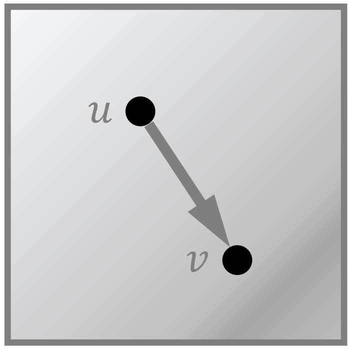
HOMOGENEOUS SPACES

Homogeneous Spaces

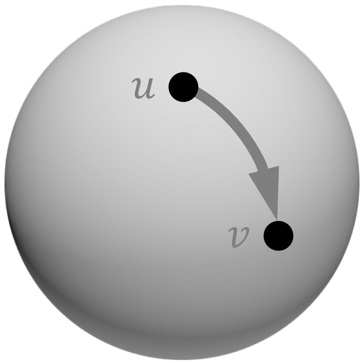
- Consider a domain Ω with a group G acting on it $G \times \Omega \to \Omega$
- The action of G is called **transitive** if for any $u, v \in \Omega$ there exists $g \in G$ such that

$$gu = v$$

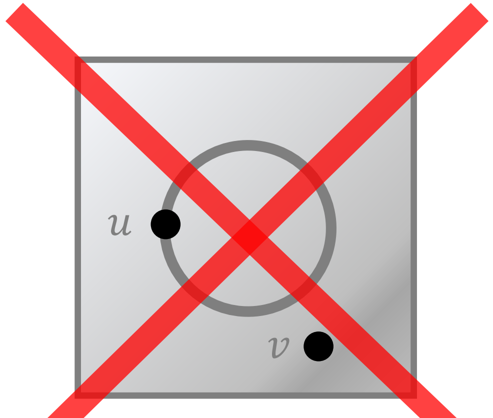
- In this case, Ω is said to be **homogeneous** for G



Plane + Translation



Sphere + SO(3)



Plane + SO(2)

Stabiliser Subgroup

- Let Ω be homogeneous for G
- The **stabiliser subgroup** of $u \in \Omega$ is $H_u = \{g \in G \;\text{ s.t. }\; gu = u\}$

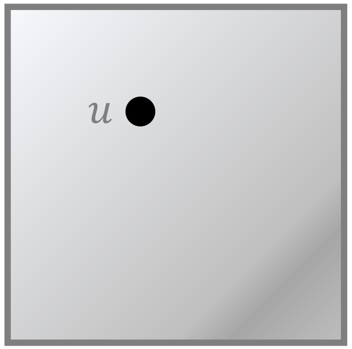
Note: we say that H is a subgroup of $(G, \circ)$, denoted $H \leq G$, if $(H \subset G, \circ)$ is also a group with the same operation

Exercise 1: show H_u is indeed a subgroup of G

Exercise 2: show that the stabiliser at any point is the same, i.e., $H_u \cong H_v$. In an homogeneous space, "every point looks the same"

Stabiliser Subgroup

- Let Ω be homogeneous for G
- The **stabiliser subgroup** is $H = \{g \in G \ \text{ s.t. } \ gu = u\}$
- A homogeneous space is *completely characterised* by G and H



$\Omega = G = \mathbb{R}^2$
$H = \{\text{id}\}$

$\Omega = \mathbb{R}^2, G = \text{SE}(2)$
$H = \text{SO}(2)$

$\Omega = S^2, G = \text{SO}(3)$
$H = \text{SO}(2)$

Cosets & Quotients

- Let G be a group and $H \leq G$ a subgroup
- A (left) **coset** of H in G is $gH = \{gh \text{ s.t. } h \in H\}$ for $g \in G$

Exercise 1: Prove that for any $h \in H$ and $g \in G$, we have $ghH = gH$.

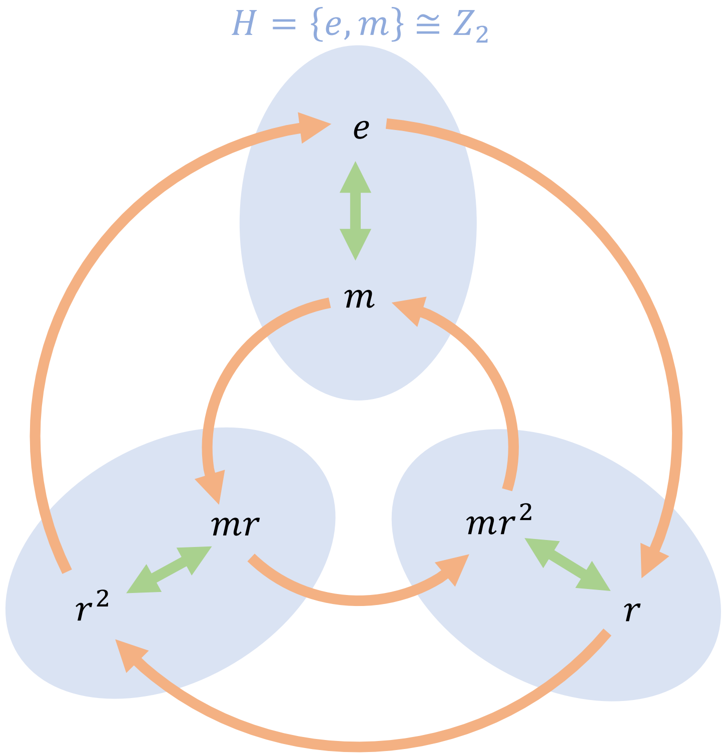
Exercise 2: Prove that if $gH = g'H$, then there exists $h \in H$ such that $gh = g'$.

Exercise 3: Prove that cosets gH and $g'H$ are either disjoint or identical, i.e., cosets partition the group.

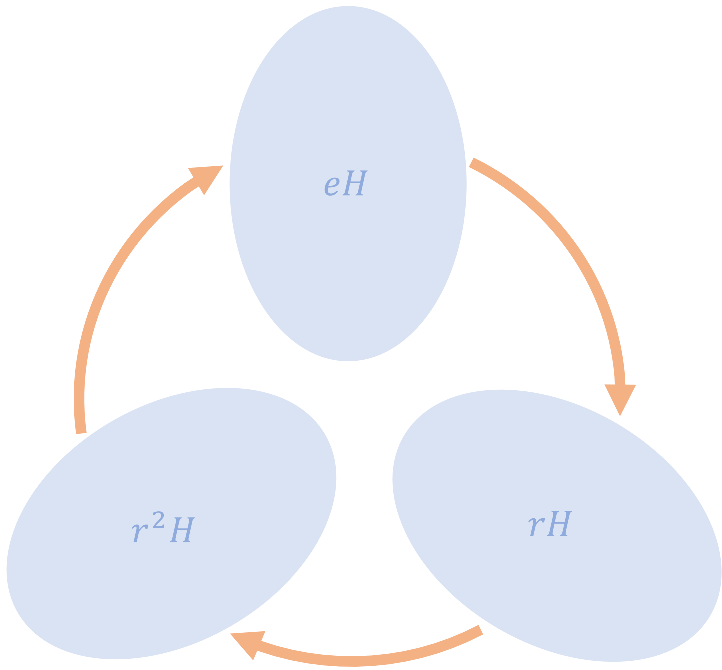
Cosets & Quotients

- Let G be a group and $H \leq G$ a subgroup
- A (left) **coset** of H in G is $gH = \{gh \text{ s.t. } h \in H\}$ for $g \in G$
- Cosets partition the group (= *equivalence classes* of group elements)
- The set of cosets G/H is called the **quotient space**
- The group G acts on G/H as $g(aH) = (ga)H$

Cosets & Quotients

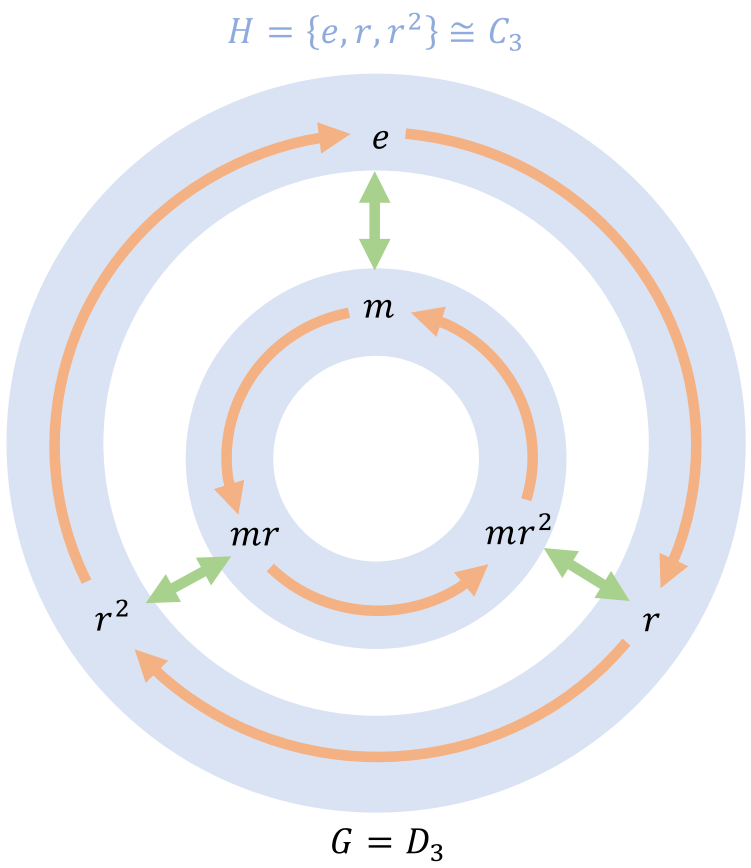


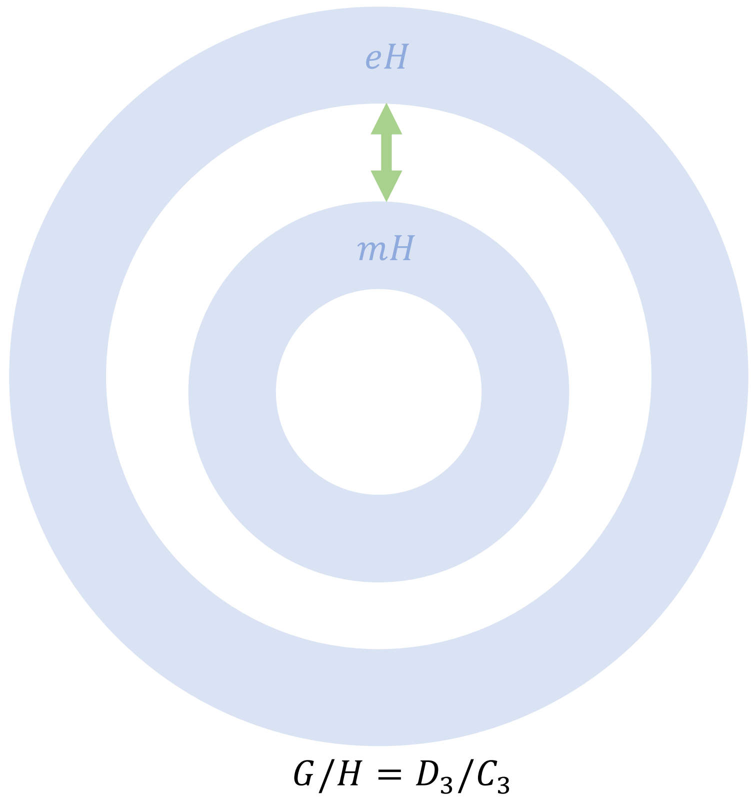
$G = D_3$



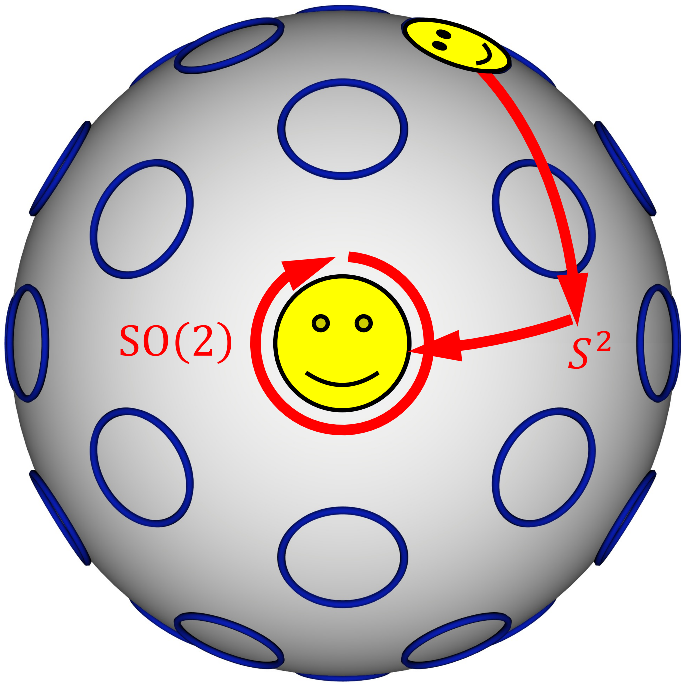
$G/H = D_3/Z_2$

Cosets & Quotients





Cosets & Quotients



$$S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$$

Note: We can view $\mathrm{SO}(3)$ as a bundle with base space S^2 and a fiber $\mathrm{SO}(2)$

Homogeneous Spaces ≅ Quotient Spaces

Quotient space G/H is homogeneous for G with stabiliser H.

Exercise: prove.

Proof:

- To show that G/H is homogeneous for G, let $u = aH$ and $v = bH$. Then $ba^{-1}u = v$, i.e., the action of G is transitive ($\exists ba^{-1} \in G$ transforming u into v).
- To show that H is the stabiliser, we choose the point $eH \in G/H$. Then, for any $h \in H$ we have
$$h(eH) = (he)H = eH.$$

Homogeneous Spaces $\cong$ Quotient Spaces

Quotient space G/H is homogeneous for G with stabiliser H.

Exercise: prove.

Let Ω be a homogeneous space for G with stabiliser H_u of some point u.
Then, $\Omega \cong G/H_u$ and u corresponds to eH.

Exercise: prove.

Example: Permutations

- Let $\Omega = \{1, \dots, n\}$ and S_n permutation group
- Ω is a homogeneous space for S_n
- Stabiliser group: pick a point $1 \in \Omega$. The subgroup $H_1 \leq S_n$ leaving 1 invariant is the group of permutations of $\{2, \dots, n\}$ equivalent to S_{n-1}

$$\begin{bmatrix} 1 & \\ & S_{n-1} \end{bmatrix}$$

  - $\Omega \cong S_n / H_1$.
  - The cardinality $|\Omega| \cong |S_n / H_1| = n$.
  - The action of S_n on S_n / H_1 is the action of S_n on Ω.

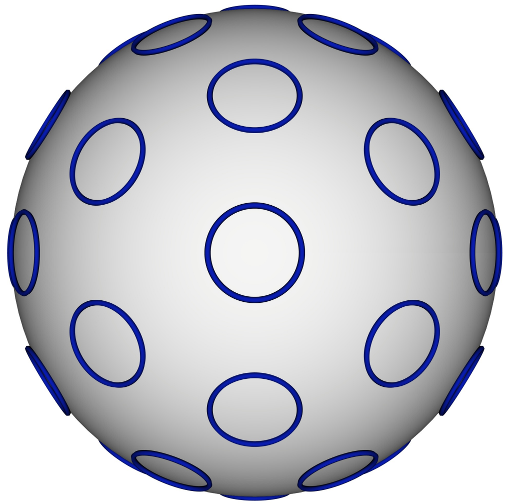
INVARIANT FUNCTIONS

H-invariant functions on $G \cong$ functions on G/H

- A function $f: G \to \mathcal{C}$ is **right H-invariant** if

$$f(gh) = f(g) \text{ for all } h \in H \text{ and } g \in G$$

Let $f: G \to \mathcal{C}$ be right H-invariant. Then, there exists a function $\bar{f}: G/H \to \mathcal{C}$ such that $f(g) = \bar{f}(gH)$.



A right SO(2)-invariant function on SO(3), i.e. one that is constant on the circles, is a function on the sphere $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$

Convolution on Spheres and Homogeneous Spaces

- **Spherical convolution**

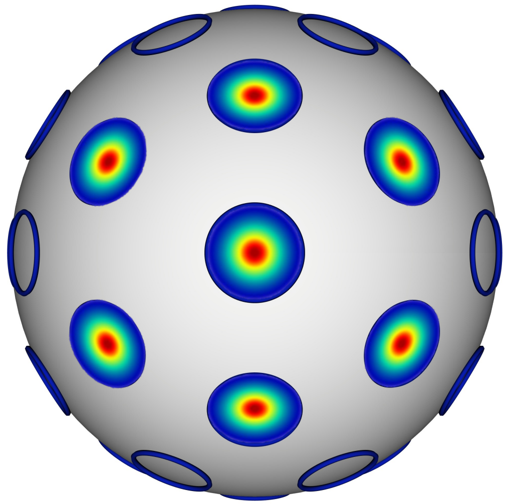
$$\star\colon \mathcal{X}(S^2)\times\mathcal{X}(S^2) \to \mathcal{X}(S^2)$$

- …is equivalent to a convolution

$$\star\colon \mathcal{X}(S^2)\times\mathcal{X}(S^2) \to \mathcal{X}\big(\mathrm{SO}(3)\big)$$

  that is $\mathrm{SO}(2)$-**invariant**

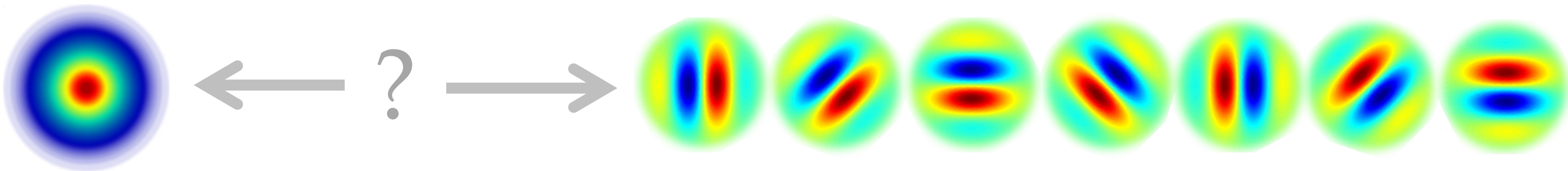
- Filters must be $\mathrm{SO}(2)$-**symmetric** (or **isotropic**)



A right $\mathrm{SO}(2)$-invariant function on $\mathrm{SO}(3)$ is a function on the sphere $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$

From Scalar to Regular

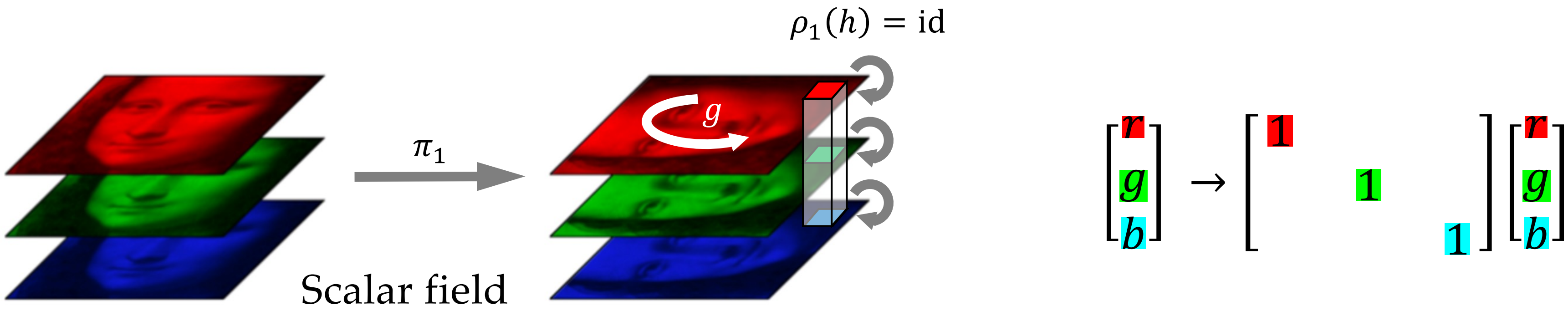
Constrained filters
Fields on G/H transforming according to
“induced representations”



Isotropic filters
Scalar fields on G/H

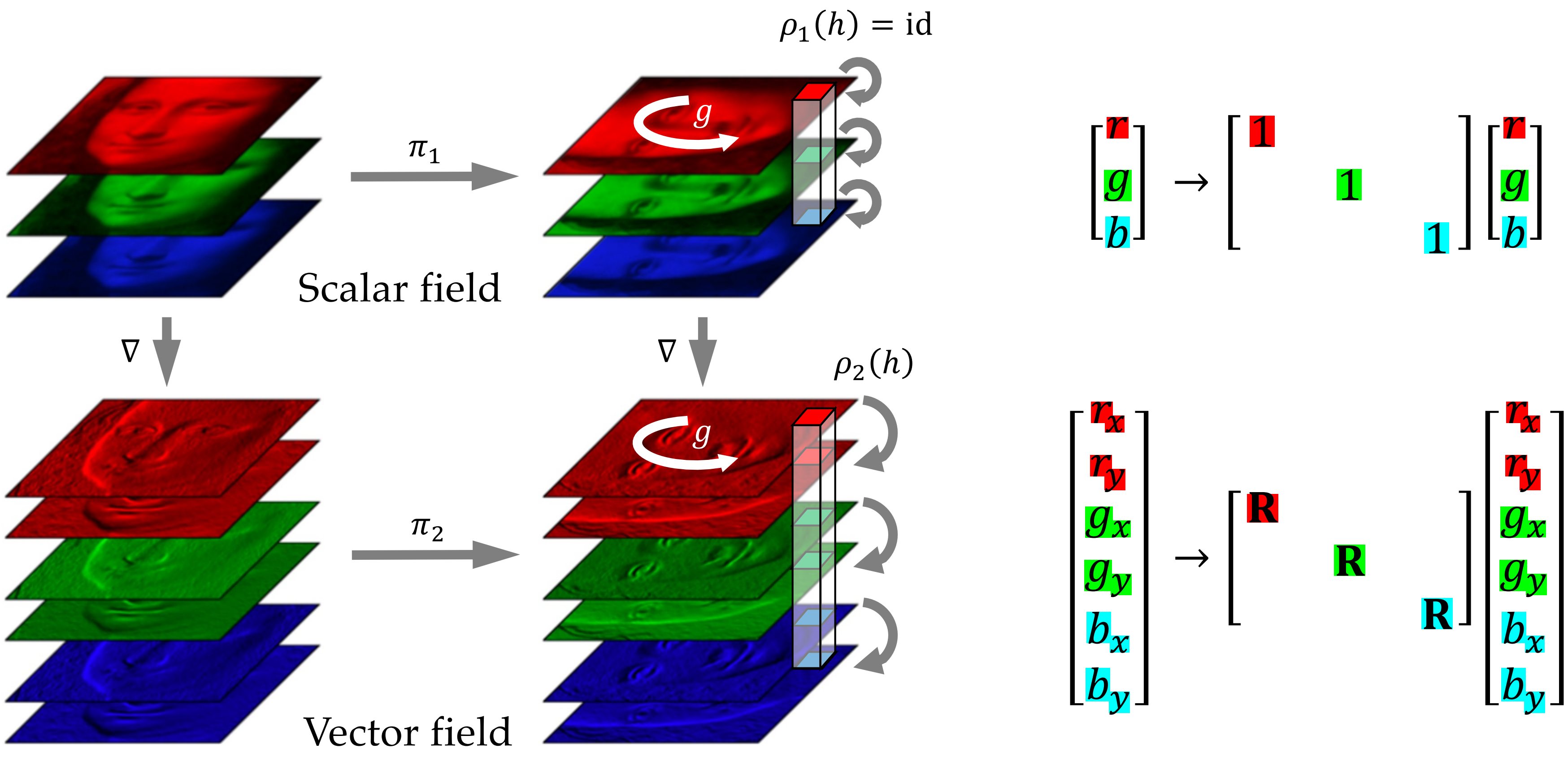
Unconstrained filters
Scalar fields on G ≅ Regular fields on G/H

Feature Fields



- Affine group: $G = (\mathbb{R}^2, +) \rtimes H$
- Stabiliser: $H = \mathrm{SO}(2)$ *"local symmetries"*
- ρ-**feature field** $x: \mathbb{R}^2 \to \mathbb{R}^C$
- ρ = representation of H acting on individual pixels ($\mathbb{R}^C$) depending on the type of the field (scalar, vector, tensor, etc.)
- $\pi = \mathrm{ind}_H^G \rho$ is **induced representation** describing how ρ-feature fields transform

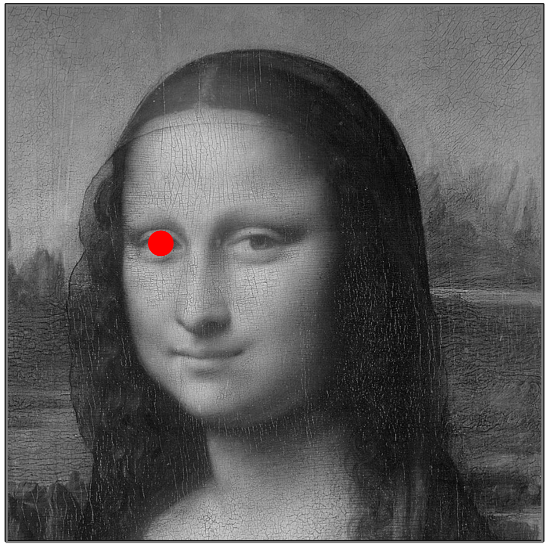
Feature Fields



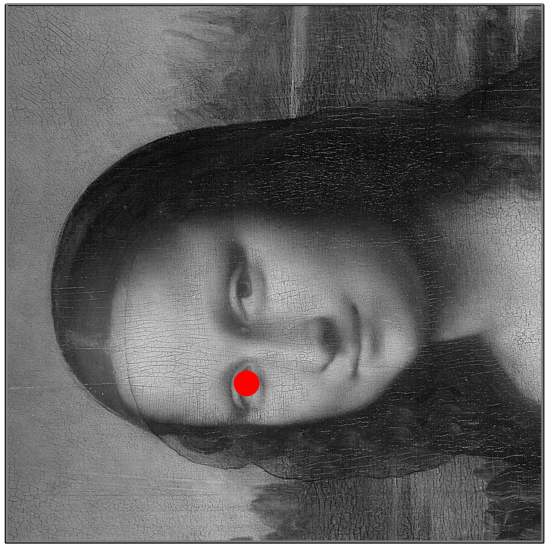
Feature Fields

$$\begin{bmatrix} r_x \\ r_y \\ g_x \\ g_y \\ b_x \\ b_y \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & & & & \\ \sin\theta & \cos\theta & & & & \\ & & \cos\theta & -\sin\theta & & \\ & & \sin\theta & \cos\theta & & \\ & & & & \cos\theta & -\sin\theta \\ & & & & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ g_x \\ g_y \\ b_x \\ b_y \end{bmatrix}$$

Transforming Scalar Fields

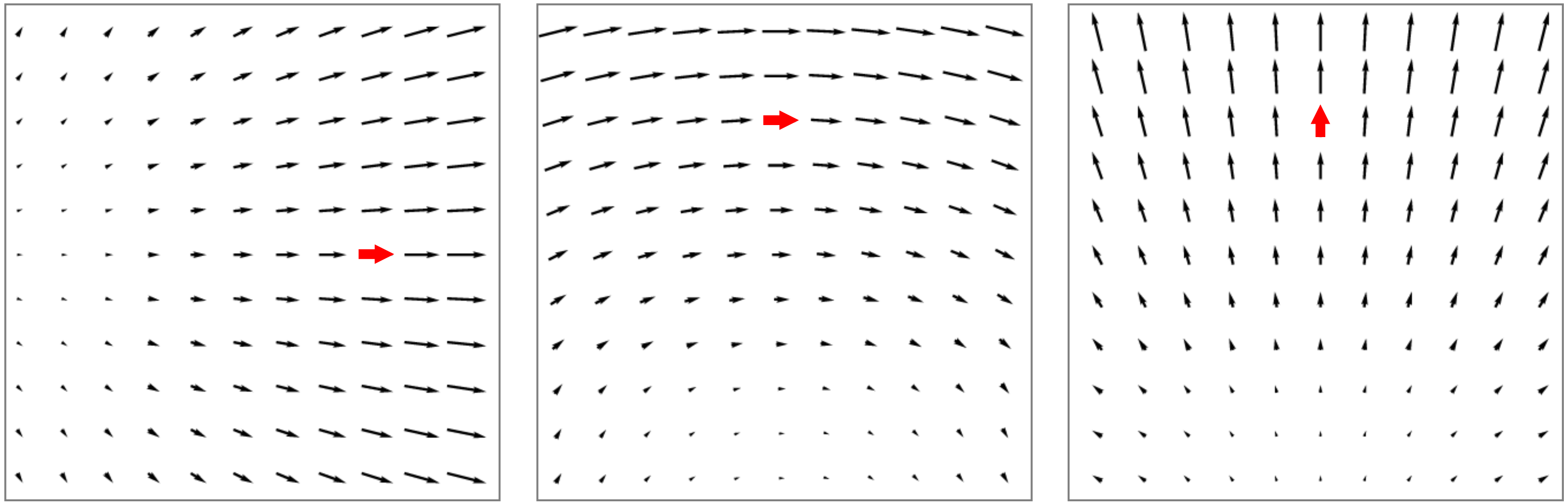


$x(u)$



$x(g^{-1}u)$

Transforming Vector Fields



$x(u)$ $\quad$ $x(g^{-1}u)$ $\quad$ $\rho(g)x(g^{-1}u)$

"Convolution is all you need"

Theorem: any linear equivariant map between induced representations is a convolution with a *steerable filter*.

- Roto-translation equivariant CNNs on the plane
- Let $G = \mathrm{SE}(2)$, $H = \mathrm{SO}(2)$ and $\mathrm{G/H} = \mathbb{R}^2$
- Equivariant convolution ("**steerable CNN**")

$$(x \star \psi)_k(u) = \int_{\mathbb{R}^2} \sum_{c=1}^{C} x_c(v)\psi_{kc}(v-u)\,\mathrm{d}v$$

where

- Feature field $x: \mathbb{R}^2 \to \mathbb{R}^C$
- **Steerable filter** $\psi: \mathbb{R}^2 \to \mathbb{R}^{K\times C}$ satisfying $\psi(R^{-1}u) = \rho_2^{-1}(R)\psi(u)\rho_1(R)$

Kondor, Trivedi 2018l Cohen et al. 2019, 2021; Bekkers 2020; Aronsson 2021

Neural Network Architectures

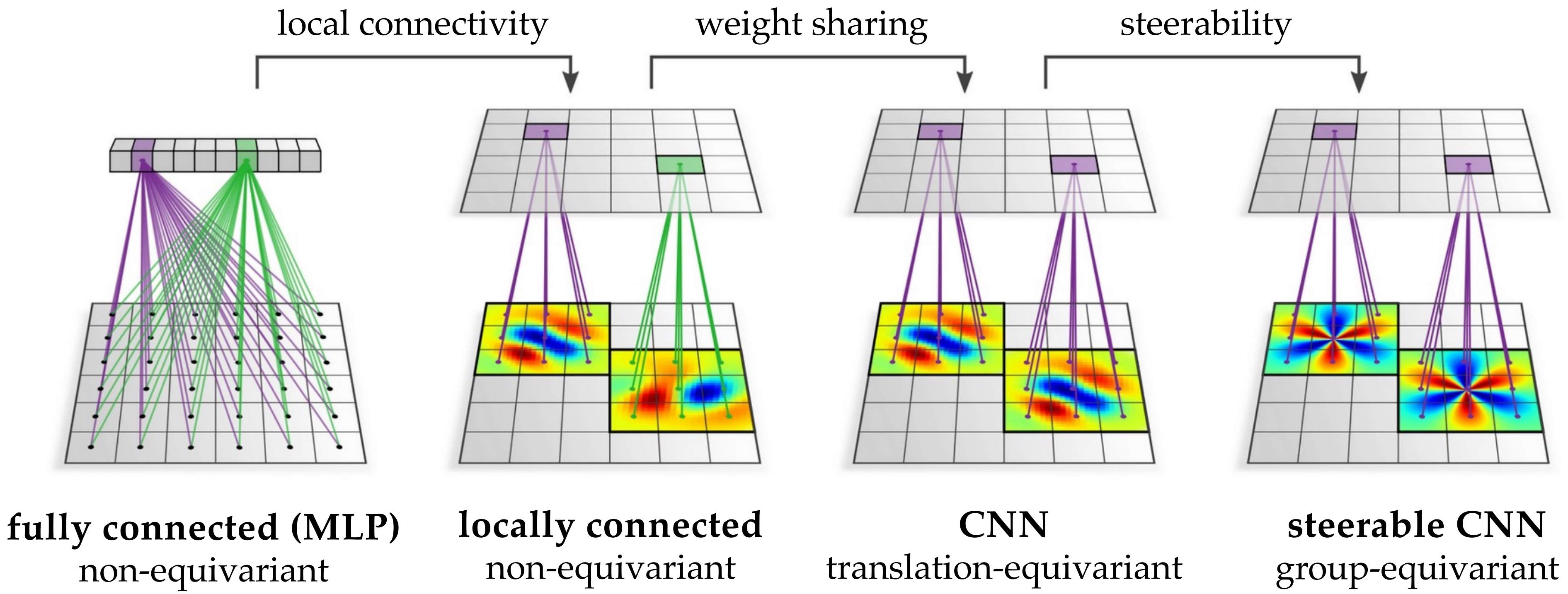


Figure: M. Weiler 2021

Takeways

- Convolution = transform + match, can be defined on *homogeneous spaces* that have “global” symmetry structure (=the group *acts transitively*)
- Homogeneous spaces are equivalent to quotient spaces, where we ”factor out” the *stabiliser group*
- *Group convolutions* are universal linear equivariants
- Next lecture: application of these ideas to Geometric Graphs and even more general notions of Convolutions and Fourier Transforms on Groups

Key Concepts

- Group convolution
- Homogeneous spaces
- Quotient spaces
- Stabiliser group
- Steerable filters

Main References

- M. Bronstein et al., Geometric deep learning, *arXiv:2104.13478, 2021*. Section 4.3 “Groups and Homogeneous Spaces” and Section 5.2 “Group equivariant CNNs”
- T. Cohen et al., A general theory of equivariant CNNs on homogeneous spaces, *NeurIPS* 2019. Convolutions on homogeneous spaces and all the group-theoretical background
- M. Weiler, Group-equivariant CNNs, First Italian Summer School on Geometric Deep Learning, Pescara 2022. Convolutions on homogeneous spaces and much more